

Singular axes in absorbing crystals

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1 Determination of the waves propagating in an absorbing crystal

2 Generic propagation directions

We consider the propagation of light in an absorbing, non-chiral crystal. A monochromatic wave with displacement vector $D \sim \exp(-i\omega t)$ with its wavevector chosen as the z-axis, is described by the fundamental equation

$$D_{tt} = c^2 A D_{zz},$$

where $A = \epsilon^{-1}$ is the inverse dielectric tensor projected onto the xy-plane. A is a complex symmetric 2×2 matrix, the imaginary part being positive semidefinite (non-active medium). The latter condition guarantees that all solutions are damped or at least non-decaying along the z-axis.

Using monochromacy

$$-\omega^2 D = c^2 A D_{zz}.$$

We redefine $A = c^2 A / \omega^2$ so that

$$A D^{(2)} + D = 0.$$

Normal solutions are obtained as

$$D = d \exp(ik_{1/2} z)$$

the polarisation fulfilling

$$(-Ak_{1/2}^2 + 1)d = 0,$$

or

$$(A - v_{1/2}^2)d = 0,$$

with $v_{1/2} = 1/k_{1/2}$. This is an eigenvalue equation for $v_{1/2}^2$ whence

$$\det(A - v_{1/2}^2) = 0,$$

$$(a_{11} - v_{1/2}^2)(a_{22} - v_{1/2}^2) - a_{12}^2 = 0,$$

or

$$v_{1/2}^4 - v_{1/2}^2(a_{11} + a_{22}) + a_{11}a_{22} - a_{12}^2 = 0,$$

with solutions

$$v_{1/2}^2 = \frac{a_{11} + a_{22}}{2} \pm \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}^2}.$$

The corresponding eigenvectors $d_{1/2}$ are

$$d_1 = \begin{pmatrix} -a_{12} \\ \frac{a_{11}-a_{22}}{2} - \sqrt{\left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}^2} \end{pmatrix}$$

and

$$d_2 = \begin{pmatrix} \frac{a_{11}-a_{22}}{2} - \sqrt{\left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}^2} \\ a_{12} \end{pmatrix}.$$

It is clear that d_2 is obtained from d_1 by a 90 degree rotation. So they have the same main axis ratio and sense of rotation. The two vectors are generically not orthogonal to each other (mind complex conjugation!)

$$d_1^+ d_2 \neq 0.$$

2.1 Singular axes

The vanishing of the discriminant

$$\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}^2 = 0$$

characterizes the singular axes. The velocities are doubly degenerate

$$v_{1/2}^2 = v^2 = \frac{a_{11} + a_{22}}{2}$$

We set

$$\lambda = \frac{a_{11} - a_{22}}{2}$$

and due to the vanishing of the discriminant

$$a_{12} = \pm i\lambda.$$

Hence

$$A = v^2 1 + \lambda(\sigma_z \pm i\sigma_x),$$

introducing the Pauli matrices. The situation becomes more clear upon a rotation with

$$R_x(\pi/2) = \cos(\pi/4) + i \sin(\pi/4)\sigma_x = \frac{1}{\sqrt{2}}(1 + i\sigma_x)$$

as the matrix A gets triangular form

$$\tilde{A} = R_x(\pi/2)AR_x(-\pi/2) = v^2 1 + \lambda(\sigma_y \pm i\sigma_x).$$

From the triangular form, it is immediately clear that the matrix is singular. From now on we drop the tilde. Based on the general theory of linear differential equations with constant coefficients and multiple roots of the characteristic polynomial we make following ansatz for the (also rotated) vector D

$$D = \exp(ikz) \begin{pmatrix} p_1 + p_2 z \\ q_1 + q_2 z \end{pmatrix}$$

with second derivative

$$D^{(2)} = -k^2 D + 2ike^{ikz} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}.$$

From

$$AD^{(2)} + D = 0$$

The solution with $a_{12} = +i\lambda$ we get

$$\begin{pmatrix} 0 & 0 \\ -2\lambda/v^2 & 0 \end{pmatrix} \begin{pmatrix} p_1 + p_2 z \\ q_1 + q_2 z \end{pmatrix} + 2ik \begin{pmatrix} v^2 & 0 \\ -2\lambda & v^2 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = 0$$

Apparently, there are no conditions on q_1 , which parametrizes the regular solution. As the equation has to hold for arbitrary values of z ,

$$\begin{pmatrix} 0 & 0 \\ -2\lambda/v^2 & 0 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = 0,$$

hence $p_2 = 0$. The last remaining condition yields,

$$-2\lambda/v^2 p_1 + 2ikv^2 q_2 = 0.$$

Therefore, the general solution is

$$\exp(ikz) \begin{pmatrix} p_1 \\ q_1 - i\lambda k^3 p_1 z \end{pmatrix}.$$

After rotating back

$$D(z) = R_x(-\pi/2)D = \exp(ikz) \begin{pmatrix} p_1 \\ q_1 - i\lambda k^3 p_1 z \end{pmatrix} = \frac{1}{\sqrt{2}}(1-i\sigma_x) \exp(ikz) \begin{pmatrix} p_1 \\ q_1 - i\lambda k^3 p_1 z \end{pmatrix}.$$

The $\sqrt{2}$ factor may be dropped, as the solution is homogeneous,

$$D(z) = \exp(ikz) \begin{pmatrix} (1 - \lambda k^3 z)p_1 - iq_1 \\ q_1 - i(1 + \lambda k^3 z)p_1 \end{pmatrix}.$$

Specifically at $z = 0$ the regular solution

$$D_{\text{reg}}(0) = \begin{pmatrix} -iq_1 \\ q_1 \end{pmatrix}$$

is a circularly polarized wave and retains its polarisation. The other wave has initially the opposite polarisation

$$D_{\text{irreg}}(0) = \begin{pmatrix} p_1 \\ -ip_1 \end{pmatrix}$$

Asymptotically, the irregular solution approaches the same polarisation as the regular one, but it is 90 degrees out of phase and is more intense due to the additional r dependence. Due to the r -dependence, the irregular solution seems to increase, but actually the solution is always decreasing due to the damped exponential.